DEPARTMENT OF NATIONAL DEFENCE
CANADA

OPERATIONAL RESEARCH DIVISION

DIRECTORATE OF OPERATIONAL RESEARCH (JOINT & LAND)

ORD PROJECT REPORT PR-9707

LOGAN (SPARING) V1.0: MATHEMATICAL THEORY

(Release 1.0)

by

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JUNE 1997

OTTAWA, CANADA

National Defence Défense nationale

102-02339/

OPERATIONAL RESEARCH DIVISION

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OTTAWA, ONTARIO JUNE 1997

ABSTRACT

The Directorate of Logistics Analysis (D Log A) has developed a new computer model known as the LOGistics ANalyzer (LOGAN) that integrates three Logistic Support Analysis (LSA) functions; sparing, level of repair analysis, and life cycle costing under a single shell. The SPARING Module is primarily designed to determine the optimal allocation and quantity of repairable spares within a Maintenance Organization to achieve a desired system goal at minimum cost. The Module can also be used to reallocate existing stock or to determine any additional spares are required in a reprovisioning exercise. The primary objective of this document is to provide a mathematical description of the SPARING Module of the LOGAN program for those who require an in-depth knowledge of the model. The methodology, assumptions, and limitations of the model are described.

RÉSUMÉ

La Direction d'Analyse logistique (DAL) a développé un nouveau programme d'ordinateur qui s'appelle LOGAN. LOGAN intègre trois fonctions de soutien logistique: l'analyse du remplacement des pièces, l'analyse du niveau de réparation, et l'analyse du coût du cycle de vie. Le module d'analyse du remplacement des pièces (SPARING) est tout d'abord conçu pour déterminer l'allocation et la quantité optimales de pièces réparables à l'intérieur d'une organisation d'entretien afin d'acquérir un but désiré pour le système à un coût minimal. Le module peut aussi être utilisé pour répartir le stock existant ou pour déterminer quelle pièces de remplacement additionnelles sont requises dans un exercise de réapprovisionnement. L'objectif premier de ce document est de fournir une description des mathématiques du module SPARING de LOGAN pour ceux qui doivent avoir une connaissance approfondie du modèle sont décrites.

EXECUTIVE SUMMARY

The Directorate of Logistics Analysis (D Log A) has developed a new computer model known as the LOGistics ANalyzer (LOGAN) that integrates three Logistic Support Analysis (LSA) functions under a single shell: sparing, level of repair analysis, and life cycle costing. The program has a graphical user interface to assist users in running the LSA Modules and a database management system which controls the project databases and allows input databases to be created from existing Logistic Support Analysis Records (LSARs). This tool is intended for users performing LSA tasks and is currently being used by engineers, Project Management Integrated Logistic Support Managers (PM ILSMs), Initial Provisioning (IP), Life Cycle Material Managers (LCMMs), and DND contractors.

The SPARING Module determines the optimal allocation and quantity of **repairable spares** within a Maintenance Organization to achieve a desired system goal at minimum cost. This selection and placement of spares is optimal in that they maximize the availability of the prime equipment (PE) for the given budget under a given set of asumptions. The Module can also be used to allocate or reallocate existing stock or to determine what additional spares are required in a reprovisioning exercise. The program allows a user to choose from a variety of performance indicators to measure system effectiveness and is capable of dealing with both multi-indentured and multi-echelon sparing problems.

The primary objective of this document is to provide a mathematical description of the SPARING Module of the LOGAN program for those who require an in-depth understanding of the model. The more sophisticated user will also gain helpful insights into modelling non-standard scenarios and evaluating model results. The methodology, assumptions, and limitations of the program are described. This review of the mathematics underlying LOGAN (Sparing) is a synthesis of various documents produced over the several years of evolution of the model. The lack of a formal configuration management approach in this development has left a poor collection of documentation. These documents, some of which were formal reports, others simply notes that were found, were in general neither complementary nor complete. Where gaps in explanation were found to exist, this report infills with the development of the necessary theory and equations. Explanations that are in conflict or are insufficiently substantiated are highlighted for further work.

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LIST OF ABBREVIATIONS

LOM	Line of Maintenance
LRU	Line Replaceable Unit (Indenture = 1)
LSAR	Logistics Support Analysis Record
MDT	Mean Down Time
METRIC	Multi Echelon Technique for Recoverable Item Control
MOD	Module (Indenture = 3)
MTBM	Mean Time Between Maintenance
MTBF	Mean Time Between Failures
PE	Prime Equipment
PLT	Procurement Lead Time
ROSAM	Recursive Optimal Spares Allocation Model
SRU	Shop Replaceable Unit (Indenture = 2)
TYPE I Failure	Failure in a Subcomponent
TYPE II Failure	Failure in the item itself

LOGAN (SPARING) V1.1:
MATHEMATICAL THEORY MANUAL

I. INTRODUCTION

HISTORICAL BACKGROUND

1. Within the Department of National Defence (DND) the Logistics Operational Research Team (LOGORT) (formerly the Directorate of Logistics Analysis (D Log A)) is responsible for conducting operational research, performing analytical studies, and developing decision support software in the areas of logistics operations, supply chain management, and logistics support activities. One area of active research and development within LOGORT is the mathematics and algorithms associated with sparing analysis.

2. Sparing analysis can be defined as the determination of the optimum allocation and quantity of spares to achieve a desired system goal. Sparing is important because it can account for 25% to 50% of the initial cost of new equipment. If it is done poorly, it can result in overspending if unnecessary spares are bought, decreased effectiveness if the wrong spares are bought, and even mission failure if the necessary spares are unavailable. Spares are divided into repairable and non-repairable items which are often referred to as repairables and consumables. Repairable items are usually high-priced, low demand items and can account for roughly 80% of the spares cost for new equipment. Non-repairable items are managed differently and sparing data for these items is normally unavailable. Sparing addressed here is that of repairable items only.

3. D LOG A began to explore and use sparing optimization models in the mid 1970's. It obtained the METRIC (Multi-Echelon Technique for Repairable Item Control) and modified it to meet DND's specific needs [16]. In 1984, D Log A created the Optimal Spares Allocation Model (OSAM) based on the METRIC and Mod-METRIC models developed by the RAND Corporation. Many enhancements were made to the original model as it was transformed into the Recursive Optimal Spares Allocation Model (ROSAM) and to its current status as the LOGistics ANalyzer sparing module known as LOGAN(Sparing). A historical overview of the development of the sparing model within D Log A can be found in [1].

4. The sparing model developed by D Log A was primarily designed to determine the optimal allocation and quantity of repairable spares for a new system but it can also be used to allocate existing undistributed stock or to determine what additional spares are required in a reprovisioning exercise. The program allows a user to choose from a variety of performance indicators to measure system effectiveness and is capable of dealing with both multi-indentured and multi-echelon problems.

AIM

5. The primary objective of this report is to describe and document the mathematical theory that is used in the sparing Module of the LOGAN program. In the sequence of evolution from METRIC/MOD-METRIC to OSAM, ROSAM and the current LOGAN (Sparing), changes to the original program were not always documented and existing documentation varies in level of detail and terminology. A secondary objective of this report is to identify and fill the gaps in our knowledge of the model as well as ensure its mathematical integrity. This is an important issue as the model was evolved over many years by several different analysts and in view of the sometimes sparse and often incomplete documentation, inconsistent changes could have been easily introduced. This document will also identify areas of shortcomings which will be addressed in future developments. The intended readers are the LOGORT analysts who will continue to progress and manage the model development. The report starts with the description of the maintenance organizational concepts used in the CF as well as the performance measures of effectiveness used. It then discusses the mathematical concepts required before elaborating on the mathematical details.

II. THE MAINTENANCE ORGANIZATION

THE MAINTENANCE PROCESS

6. The purpose of a maintenance organization is to maintain the operational availability of the equipment being used by the operational units. The maintenance organization consists of the maintenance sites which supply and repair the spare items of this "prime equipment" (PE). These maintenance sites are arranged in echelons or lines of maintenance based on location and repair capability.

7. First line sites are located with the operational units where PE fail, generating demands for spare items. The first line sites are responsible for identifying, and replacing failed components, which are known as line replaceable units (LRUs). If this maintenance site does not have a working LRU in stock, it must either obtain a spare from another site, or repair the broken one. If the failed LRU can be disassembled into subcomponents, known as a shop replaceable unit (SRU), it will be broken down to identify the failed subcomponent. This disassembly continues until the failed item is identified and repaired. If the failed item cannot be identified or repaired at the first line site, the item is shipped to a second line site that has a greater capability. If the second line site does not have the required capability, the item is shipped to either a third line site, or a contractor for repair. If the item is damaged beyond repair, it is discarded and a new one must be procured. Similarly, if a second line site has no spares in stock when it receives an order from a first line site, it must order a spare from a third line site, or possibly a contractor.

MEASURING THE EFFECTIVENESS USING BACK ORDERS

8. There are many ways to measure the effectiveness of the maintenance organization because it is a function of many different factors such as the repair capability, the stock on hand, and the time needed to acquire a spare at each site. For example, someone in supply may be trying to meet their goal of satisfying 90% of all orders received, whereas someone in operations may be trying to ensure that 75% of a fleet is operational on any given day. The measure of effectiveness (MOE) to choose is the one that best demonstrates whether the specified performance target is being met.

9. One common measure of effectiveness is the expected number of back orders. A back order is said to exist when a maintenance site is unable to satisfy a need for a spare. The expected number of back orders can be calculated by adding the number of days any item is back ordered over a representative time period, and dividing by the number of days in that time period. The developers of the METRIC model found that, from a mathematical point of view, using back orders as the measure of effectiveness was preferable to other measures because back orders form a convex function by definition and therefore a solution can be found. In addition, it seems that minimizing the expected number of back orders provides good results for other measures of effectiveness, but the converse is not necessarily true [2]. The sparing model developed by D Log A is also based on this concept. The other measures of effectiveness supported by the model are discussed later in this report.

III. BASIC MATHEMATICAL CONCEPTS

INTRODUCTION

10. In general, modelling physical systems mathematically requires some degree of simplification. The following concepts are therefore used in describing the maintenance structure and process as well as the equipment structure.

INDENTURE SYSTEM

11. A multi-indentured system is used by the model to represent the item breakdown of the PE as shown in Figure 1. In this hierarchical relationship, the PE is broken down into Line Replaceable Units (LRUs) which form the first indenture level of the PE. These LRUs may contain sub-components called Shop Replaceable Units (SRUs) which form the second

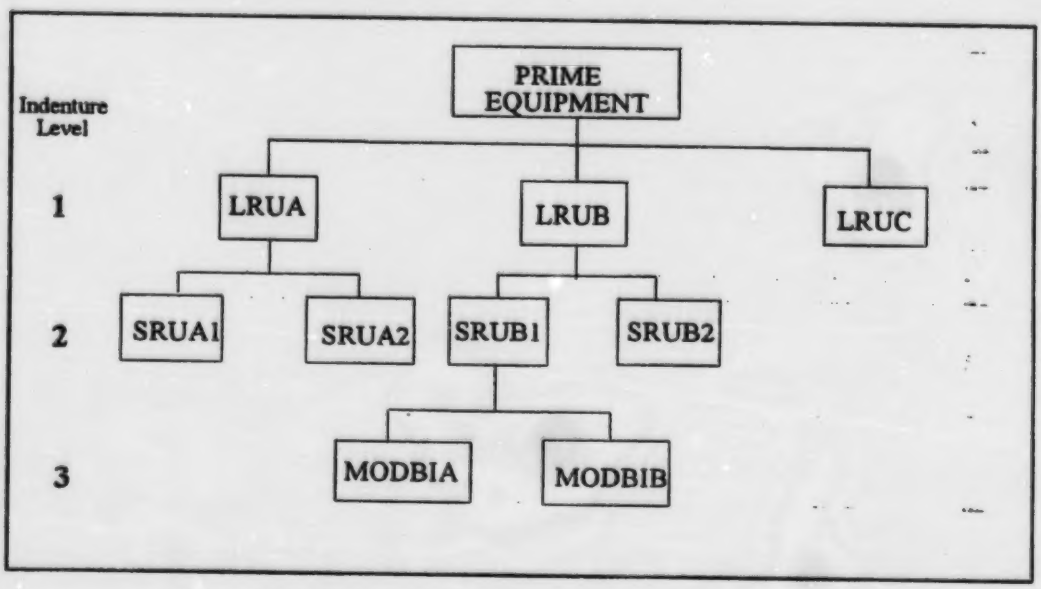

Figure 1: Multi indenture Item Breakdown

indenture level of the PE. An SRU is often referred to as a "child" of its "parent" LRU. This "parent-child" relationship extends downwards from component to sub-component until the last item that can be repaired by the maintenance organization is reached. The indenture level of an item is always one higher than that of its parent item.

ECHELON SYSTEM

12. A multi-echelon system is used by the model to represent the maintenance organization as shown in Figure 2 . Maintenance sites are grouped into echelons or lines of maintenance (LOM), each with its own repair capability. Higher echelon sites support lower echelon sites by providing working items from their inventory of spares, repairing broken items, or by requesting spares from the next higher echelon. The first line sites form the lowest echelon where the maintenance process begins. The single maintenance site at the top of the echelon structure represents the last source of spare items within the maintenance organization. Broken items arriving from lower echelons that cannot be repaired at this site are replaced by procuring new items from the manufacturer. The time required to obtain a new item from the manufacturer is known as the procurement lead time (PLT). For lower echelons, the time required to obtain a replacement item from a higher echelon includes the time taken at the higher echelons to find a spare in the system if they are out of stock and the time to process the order and ship it, the latter being referred to as the Order and Shipping Time in Figure 2. Spares "pipe-lines" are established for the flow of failed and replacement items between each site and its support site. This is an idealized logistics system because spare items are assumed to move only in a vertical direction in the echelon system with each site having only one higher echelon site to draw upon. For more information on multi-indentured, multi-echelon systems the reader is directed to [5].

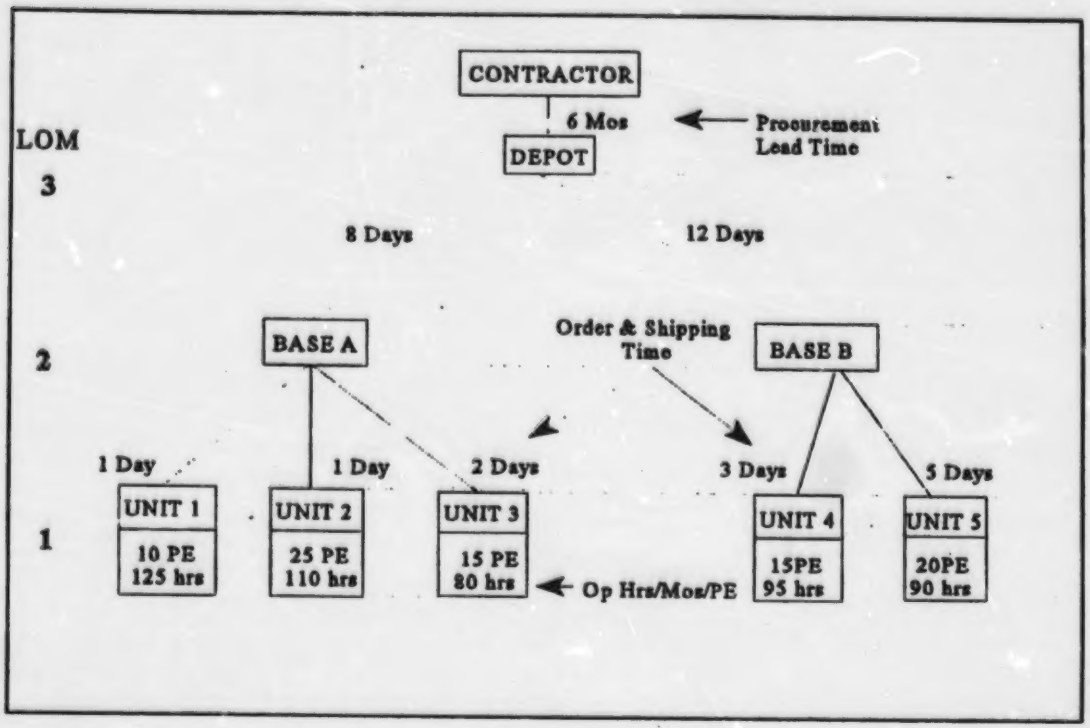

Figure 2: Multi-Echelon Maintenance Organization

(S-1,S) INVENTORY POLICY

13. Inventory systems are often characterized by an order point level S at which stock resupply is triggered for an amount that raises the stock up to a maximum level M. We refer to this as an (S,M) inventory policy. The value of S is set so as to be reasonably certain that no stock out will occur during the resupply. M is often set by identifying an economic order quantity Q by setting M = S + Q. Inventory systems for high-cost, low-demand repairable items are generally modelled according to an (S-1,S) inventory policy which translates into a one for one replacement policy. This means that a replacement item is ordered as soon as a failure occurs and that no batching of failures takes place before a repair or resupply request is made. In inventory terms, this means the economic order quantity is equal to one (Q=1) because there is a one-for-one repair. The LOGAN sparing model also assumes this policy is in effect, and that no batching occurs. For more information on (S-1,S) and other inventory policies, see [3].

PREDICTING FAILURES

14. Unfortunately, one cannot predict exactly when a given item will fail because there are many factors that can cause failure and most of them are random in nature. One must therefore refer to an inherent characteristic of the item known as its reliability which is defined as the probability that the item will perform without failure for a given time period under specified operating conditions. It is determined by testing samples of the item under the specified operating conditions and counting the number of failures over time. A probability distribution of the time between successive failures can also be made. The failure rate of the item is the number of failures that occurred in a specified time interval and is normally expressed as either the mean failure rate or its reciprocal concept the mean time between failures (MTBF). The probability density function most commonly chosen to represent the interarrival times between failures is the exponential which for line data have consistently shown to be a good model, but other distributions such as the Rayleigh or Weibull have also been used to describe failure characteristics. In many cases, a function is chosen based on a priori knowledge of similar items. The sparing model within LOGAN assumes that the distribution of interarrival times between failures is exponential. For more information on reliability, the causes of failure, and the functions used to model such phenomenon, see [4].

POISSON DISTRIBUTION

15. Processes that generate demands based on the exponential distribution are often referred to as being Poisson because it can be shown that the probability that a specific number of demands will occur in a given time period follows a Poisson distribution if the interarrival times between failures is described by an exponential distribution. The Poisson probability of having no demands in a given period is by definition equal to the reliability of the item. The probability that a specific number of spares will be required in the given period can now be determined. One basic assumption of the Poisson process is that in arbitrarily small periods of time at most one demand will occur. If the quantity of spares demanded can vary from demand to demand, then a second distribution is required to describe this characteristic. In this case, a compound Poisson distribution such as the negative binomial would represent the probability that a specific number of spares will be required in a given period. The sparing model within LOGAN considers Poisson demands only, and therefore assumes spares are demanded one at a time.

16. The Poisson distribution has proven to be a good approximation of the failure rate for many items, and it is easy to work with mathematically because it is a one parameter distribution based on the failure rate. Specific probabilities can be calculated from Equation (1) where $P(k;\lambda T)$ is the probability of having exactly k failures in a given time interval T for an item with a failure rate λ.

$$P(k;\lambda T) = \frac{e^{-\lambda T}(\lambda T)^k}{k!} \qquad (1)$$

The failure rate is assumed to be constant, with $\lambda = 1/MTBF$, and λT, the mean of the distribution, represents the expected number of demands in the given time interval T. For more information on Poisson distributions, see [3, 4].

PALM'S THEOREM

17. If a Poisson process is being used to represent demand and the mean of the repair time distribution is known, a queueing theorem of Palm's can be used to estimate the steady-state probability distribution of the number of units in repair. Palm's Theorem states that, "If demand for an item is a Poisson process with annual mean m and if the repair time for each failed unit is independently and identically distributed according to any distribution with mean time T then the steady-state probability distribution for the number of units in repair has a Poisson distribution with mean mT." (Ref 5) This theorem is one of the major building blocks of the mathematical theory behind repairable item inventory theory. It simplifies the problem enormously as it eliminates the need to measure the shape of the repair time distribution although it does imply that repair queues are infinite in length. The LOGAN sparing model was developed using Palm's theorem and therefore assumes that repair times are independent and identically distributed. For more information on Palm's theorem, see [5, 6].

LITTLE'S FORMULA

18. Little's formula also comes from queueing theory and is probably best explained using an example. Consider a bank where customers arrive and, if a teller is free, are served right away else they must stand in line. Standard performance measures are usually the steady-state values of the mean number of customers in the bank, the average number of busy tellers, the

throughput rate, and the mean time that a customer spends in the bank. The randomness of the arrivals and serving times cause fluctuations in the values of the performance measures, but the running average will always settle down to a constant or steady-state value after a sufficient number of observations. Thus, steady-state does not mean, for example, that the number of people lined up waiting for a teller in the bank is constant after a certain period of time, but that over the long term the average number of people in the line will be constant.

19. There are many relationships among the performance measures and Little's formula shown in Equation 2 links L, the mean number of customers in the system, W, the mean time that a customer spends in the system, and λ, the arrival rate.

$$L = \lambda W \tag{2}$$

20. Little's formula is a very powerful tool because it applies on a broad scale to any system that transforms input to output over time, and possesses steady state performance measures corresponding to L, λ and W. The rule can be applied to any portion of a system, or to collections of systems. In LOGAN (Sparing) it is used to transform backorders into supply delays. For more information on Little's formula, see [7, 8].

OPTIMIZATION

21. Optimization can be defined as the science of determining the "best" solutions to certain mathematically defined problems, usually models of physical systems. The form of the problem is usually that of an objective function which is to be maximized or minimized with a number of variables subject to restrictions or interrelationships which form constraints. The constraints may contain many relationships including equalities, inequalities, as well as linear and nonlinear functions. The solution is normally dictated by the type of constraints, with integer programming, linear programming, or marginal analysis techniques being used depending on the constraints. If some of the constraints are nonlinear, the problem becomes much more complicated. The objective function of the sparing model represents a large scale nonlinear integer programming problem that is usually solved via marginal analysis. LOGAN (Sparing) uses marginal analysis to optimize each LRU and all its subcomponents and uses nonlinear programing to perform trade offs between LRU's. For more information on linear and non-linear programming, see [9].

MARGINAL ANALYSIS

22. Marginal analysis refers to a technique in which the effect of a unit change in the variable being studied is examined. At each step of the algorithm only one number needs to be looked at to determine the next course of action. The marginal or incremental value provides all the information necessary to make the decision. The sparing model within LOGAN uses a marginal analysis technique to choose the next spare and the location where it will cause the largest increase in system effectiveness. For more information on marginal analysis, see [5, 6].

IV. MATHEMATICS OF LOGAN (SPARING)

THE "OPTIMAL" SOLUTION

23. In general, the LOGAN (Sparing) model was designed to determine the optimum allocation and quantity of repairable spare items by minimizing the expected number of back orders in the maintenance organization. In determining the number of back orders, the model restricts its calculations to LRU back orders at first line sites in order to satisfy the primary goal of the maintenance organization which is to maintain the availability of the prime equipment at the operational units. There are many other factors that affect availability but for simplicity must be excluded. In particular consumable spare parts affect the availability of repairables and thus the PE. There exclusion is equivalent to assuming they are always available. In addition, to find the optimal solution for just the repairable items every possible combination of spares at every location would have to be compared. For almost any real world problem, it is not feasible to calculate all possible combinations. Therefore, the approach taken is the marginal analysis technique of determining the change in back orders caused by one additional spare of every item at every location. The change in back orders is measured from the baseline established by the current distribution of spares. The one problem with marginal analysis is that, in the allocation of one item, it is seen as a scarce resource. This has a tendency to place it higher up in the system. Yet as stock builds up and is not so scarce it should be pushed down to the bases. Once the next spare has been chosen, the model then calculates the effect of the addition on similar spares at higher supporting echelon locations to determine if a reallocation of the item is more beneficial. This approach is heuristic and does not necessarily provide the optimum solution, but is a good approximation to it.

24. It must be taken into consideration however that there is a logistics cost associated with every spares decision. Therefore, the optimum solution is not necessarily the one that maximizes system performance alone, but is the one that maximizes system performance due to spares at minimum cost. The cost is normally measured in dollars, but other logistics parameters such as volume or weight can sometimes be more important. The LOGAN (Sparing) model allows a user to choose between dollars, volume, or weight as the logistics cost parameter.

OBJECTIVE FUNCTION

25. It is sometimes impossible to achieve the desired system performance goal because of budgetary or physical limitations which constrain the possible solution. In these cases, the optimal solution is to maximize system performance within the given logistics cost limit. Therefore, more specifically, the model determines the quantity and location of repairables that minimize LRU back orders at first line sites with respect to the logistics cost. This mathematical problem may be formulated as

$$\min \sum_{j=1}^{M} \sum_{B=1}^{N} B\left(S_j^B\right) \quad \textit{such that} \quad \sum_{j=1}^{m} \sum_{B=1}^{n} S_j^B C_j \leq C \qquad (3)$$

where

M = number of LRUs,

N = number of first line locations,

m = total number of parts ($M \leq m$),

n = total number of locations ($N \leq n$),

$B(S)$ = expected number of back orders with stock level S,

S_j^B = number of spare j at location B,

C_j = cost of part j,

C = cost constraint.

The solution to Eqn 3 is then used to optimize system performance for a fixed budget, or to achieve a stated performance goal at minimum cost. Ideally, all components would be considered simultaneously; however, due to memory considerations, LOGAN (Sparing) calculates an optimal spares distribution for each LRU and its sub-components separately and then sorts the individual results to provide the optimal distribution of spares for the entire PE. This will be explained in more detail later on.

ASSUMPTIONS

26. The assumptions made in LOGAN (Sparing) are:

a. The PE can be described as a multi-indentured system of items as described in Section IV and that failures of these items follow a stationary Poisson process. This implies the MTBF of each item is constant through time and that multiple failures are not modelled.

b. The maintenance organization can be described as a multi-echelon system as described in Section IV and therefore there is no lateral resupply between sites of the same echelon, and no resupply from locations of a lower echelon.

c. There is a predetermined, fixed maintenance plan based on the concept that an item is broken down into components only at one echelon level (LOM) in the maintenance organization and that no demands for a subcomponent can occur until the item has been broken down. The term "breakdown echelon" shall refer to the echelon at which an item is broken down into its subcomponents.

d. The probability of failure in a given LRU is independent of all other LRU failures.

e. Failure in an item is caused by the failure of the item itself or else of at most one subcomponent if the item itself did not fail.

f. Repair starts immediately upon identification of the failed component and no batching of failures occur.

g. Repair times are independent and repair queues infinite.

h. Cannibalization from other non-operational PE is not permitted.

NOTATION

27. Given a multi-echelon, multi-indentured problem, a method is needed to distinguish the variables for each item at every location. The notation adopted in this report is to use subscripts while referring to items and superscripts to refer to location. Using this notation, λ_j^B refers to the demand rate of item j at site B. To describe the relationship between sites, a superscript A refers to the support site (parent) of B while a superscript C refers to a supported (child) site of B. Similarly, a subscript i refers to the parent of item j while a subscript k refers to a child (subcomponent) of item j.

28. Since LOGAN (SPARING) does not take into account non-repairable spare items, the terms "spares", "repairables", and "repairable spare items" are used interchangeably throughout this report.

DERIVATION OF BACK ORDERS EQUATION

29. With the assumptions given in the previous section, a formula for calculating the expected number of back orders for a given item at a particular location can now be developed. When an item fails, a PE is unavailable until a working item is resupplied. If the average time to resupply the item is R and λ represents the failure rate of the item, then (λR) represents the expected number of demands that should be received during the resupply time R. From Palm's theorem, the steady state probability (P) of having k demands during R is given by

$$P(k;\lambda R) = \frac{e^{-\lambda R}(\lambda R)^k}{k!} \qquad (4)$$

The probability of having S or fewer demands is found by summing the probabilities of having zero to S demands as shown below:

$$P(k \leq S) = \sum_{k=0}^{S} \frac{e^{-\lambda R}(\lambda R)^k}{k!} \tag{5}$$

This is also the probability that a stock of S spares would satisfy the expected demand during the resupply cycle and is often referred to as the availability of spares, $A(S;\lambda R)$.

30. If S+1 failures occur during the resupply time R, then S spares cannot satisfy all the demands for spare items, and a back order is said to exist. The probability of having at least one back order can be determined by summing the probabilities for k being greater than S:

$$P(k > S) = \sum_{k=S+1}^{\infty} \frac{e^{-\lambda R}(\lambda R)^k}{k!} \tag{6}$$

The expected number of back orders can be calculated by taking the weighted average of the probabilities given in Equation (6).

$$\begin{aligned} B(S;\lambda R) &= \sum_{k=S+1}^{\infty} \frac{e^{-\lambda R}(\lambda R)^k}{k!}(k-S) \\ &= \sum_{k=0}^{\infty} \frac{e^{-\lambda R}(\lambda R)^k}{k!}(k-S) - \sum_{k=0}^{S} \frac{e^{-\lambda R}(\lambda R)^k}{k!}(k-S) \\ &= \begin{cases} \lambda R - S - \sum_{k=0}^{S-1} \frac{e^{-\lambda R}(\lambda R)^k}{k!}(k-S), & \text{if } S \neq 0 \\ \lambda R, & \text{if } S = 0 \end{cases} \end{aligned} \tag{7}$$

In order to calculate the expected number of back orders using Equation (7), formulae for determining the demand rate (λ) and resupply time (R) for any item at any location are required.

31. It should be obvious by now that the whole allocation process depends heavily on the capability of computing efficiently all sorts of delays, and therefore hinges on an efficient algorithm for computing back orders. The availability and back orders equations can be expressed recursively so as to be computationally more efficient in the following manner:

$$
\begin{aligned}
A(0,\lambda R) &= 0 \\
A(S+1;\lambda R) &= A(S;\lambda R)+\frac{e^{-\lambda R}(\lambda R)^{S+1}}{S+1} \\
B(0;\lambda R) &= \lambda R \\
B(S+1;\lambda R) &= B(S;\lambda R)-1+A(S;\lambda R)
\end{aligned}
\qquad (8)
$$

DEMAND RATES (λ)

32. Although the demand or failure rate (λ) is assumed to be constant with time, it does vary from base to base because first line sites support different numbers of PE. Failures repaired at one maintenance echelon are not sent to the next echelon, if failed subcomponents are identified and replaced there. Thus there is a decomposition of the failure rates caused by the echelon maintenance structure, as well as the equipment structure. The modelling of this decomposition will now be discussed.

Type I Versus Type II Failure Fraction

33. Failures in an item may be due to failure of the item itself or due to failure of one of its subcomponents. When the item fails because of a failure in one of its subcomponents, the item is said to have a Type I failure. Type II failures are the ones within the item itself and not traced to one of its subcomponents. For example, consider an item (LRU) described as a circuit board containing two replaceable electronic chips (SRUs). If the item fails because

one of the two electronic chips failed, a type I failure has occurred. If the item fails because the circuit board is malfunctioning and the two electronic chips are both working fine then a type II failure has occurred. The type I failure fraction due to subcomponent k in item j is given by:

$$f_k = \frac{Q_k \, MTBF_j}{MTBF_k} \qquad (9)$$

where:

Q_k = quantity of subcomponent k in part j;

$MTBF_j$ = mean time between failure of part j.

and the type I failure fraction of part j would be

$$f_j = \sum_{k=1}^{m} f_k$$

where m is the number of subcomponents in item j.

34. Since the sum of all the failure fractions equals one, the Type II failure fraction can be determined from:

$$\bar{f}_j = 1 - f_j \qquad (10)$$

If the item has no subcomponents, the Type II failure fraction is one. On the other hand, if the item itself has no failures, i.e. is not a repairable item, then it would not be spared and we could conceptualize the assembly as being composed strictly of components. In particular, once the part is broken down, we deal only with its repairable components. In this way we can reduce the equipment breakdown structure found in the Logistics Support Analysis Record (LSAR) to one which is more manageable by deleting from it parts with zero or no failure rates (most parts identified in the LSAR would not be repairable). It is, however

important to realize that this procedure is more than conceptual. Though it is true that the circuit board in the example above can be considered conceptually as a subcomponent, like the chips, with its own failure rate, if it can not be purchased from the contractor without the chips, then it cannot be managed as a separate item. In the case where $\bar{f}_j > 0$, this model will establish stock of full assemblies of this item at sites above its breakdown level, even though we are modelling the item without its subcomponents at these sites. On the other hand if $\bar{f}_j = 0$, the disassembly and removal replacement times of the subcomponents must be accounted for if they are not trivial, so caution must be exercised before deleting it.

35. Before proceeding we need to define two factors or "Switches" which will permit us to aggregate all the formulae developed for the demand rates, as well as for the resupply times later, into a more compact form . Let

$$\delta_j^B = \begin{cases} 0 \text{ if item } j \text{ is NOT broken into components at site } B \\ 1 \text{ if item } j \text{ is broken into components at site } B \end{cases}$$

$$\Delta_j^B = 1 - \delta_j^B\left(1 - \bar{f}_j\right) = \begin{cases} 1 \text{ if item } j \text{ is NOT broken into components at site} B \\ \bar{f}_j \text{ if item } j \text{ is broken into components at site } B \end{cases}$$

Calculating Demand Rates

36. The demand rate λ_j^B for an item j at a site B is obtained in a recursive fashion, given the demand for its parent assembly is known. The formulae for calculating the demand rate of item j at site B are developed by considering four separate cases.

Case 1: part j is an LRU and site B is a first line unit.

If the item is an LRU at a first line location, the demand rate would be equal to the total operating hours of the LRU divided by its MTBF as shown in Equation (11a). The total operating hours of the LRU is found by taking the product of the number of prime equipment (PE), the number of LRU per equipment, and the hours of operation.

$$\lambda_j^B = \frac{E^B H^B Q_j}{MTBF_j} \tag{11a}$$

where:

λ_j^B = *number of failures of part j per day at site B*;

E^B = *number of prime equipment at site B*;

H^B = *operating hours per day per PE at site B*;

Q_j = *quantity of part j in the PE*;

$MTBF_j$ = *mean time failure for part j*.

Case 2: part j is a subcomponent of part I and site B is the breakdown level of part I.

If the item is a subcomponent, then the type I failure fraction for part j must be applied at the maintenance echelon where the parent item is broken down.

$$\lambda_j^B = f_j \lambda_i^B \tag{11b}$$

Case 3: site B is one echelon above the breakdown echelon for item j.

In this case the demand rate is found by summing the demands on the supported (child) sites of B weighted by the type II failure fraction and the percentage of failures not repaired by the supported site. In this case, failure rate is equal to:

$$\lambda_j^B = \sum_{C=1}^{N} \lambda_j^C \bar{\phi}_j^C \bar{f}_j \qquad (11c)$$

where

$\bar{\phi}_j^C$ = *fraction of failed parts j NOT repaired at site C;*

N = *number of supported sites of B;*

Case 4: for all other cases.

The demand rate at B is calculated by summing the demands on the supported sites of B weighted by the percentage of failures not repaired at the supported site:

$$\lambda_j^B = \sum_{C=1}^{N} \lambda_j^C \bar{\phi}_j^C \qquad (11d)$$

Using the switching factors defined earlier at paragraph 35 we can combine these four cases into one equation and summarize the whole in the following equation:

$$\lambda_j^B = \begin{cases} \dfrac{E^B H^B Q_j}{MTBF_j} \text{ if part } j \text{ is an LRU and site } B \text{ is a first line unit;} \\ f_j \lambda_i^B \text{ if part } j \text{ is a subcomponent of part } i \\ \quad \text{and site } B \text{ is at the breakdown echelon of part } i; \\ \sum_{C=1}^{N} \lambda_j^C \bar{\phi}_j^C \Delta_j^C \text{ for all other cases.} \end{cases} \qquad (11)$$

Before proceeding to the development of the resupply time formulae it is emphasised that λ_j^B represents the flow of part j into site B. In particular, if B is the breakdown level of this part this demand rate has not been broken down into its components. Though this could be done at this stage it is best to wait and integrate the breakdown of the part into the resupply time

part of the equations. Finally we note that these formulae for the demand rates are recursive and must begin at the highest indenture level of the equipment but at the lowest echelon level of maintenance.

WEIGHTED RESUPPLY TIME (R)

37. Palm's theorem can be extended to the multi-echelon, multi-indenture case by expressing R as a weighted resupply time instead of the average repair time, where resupply time is defined as the time required to restore a failed item to an operational ready state at the site and return it to supply. The resupply time is still a function of repair time but it is also a function of the time required to ship items between echelons, delays in obtaining a spare from a supporting echelon and the time required to remove replace or obtain any subcomponents (see Figure 3). We identify three separate components of the resupply time of item j at site B defined as:

$$R_j^B = RP_j^B + RS_j^B + RR_j^B$$

where:

RP_j^B = *average resupply time due to the part j itself at site B;*

RS_j^B = *average resupply time due to subcomponents of part j at site B;*

RR_j^B = *average time to receive a broken part j from supported sites of B.*

Before proceeding with the description of each of these components, it would be helpful to discuss Figure 3. What we are trying to determine is the time it takes to receive a failed part j and process it back into the supply system at site B. The first thing to decide is whether the part is "repairable this site" (RTS). If so then the delay is the repair time r_j^B or RS_j^B if the item is broken down at B and a failed subcomponent is repaired. If not, then it is shipped to the support site A and another part is ordered. If there is one in stock at site A then the delay is the order and ship time O^{AB}. If site A is backordered then there is a delay D_j^A. This last delay can be computed from Little's Law, namely

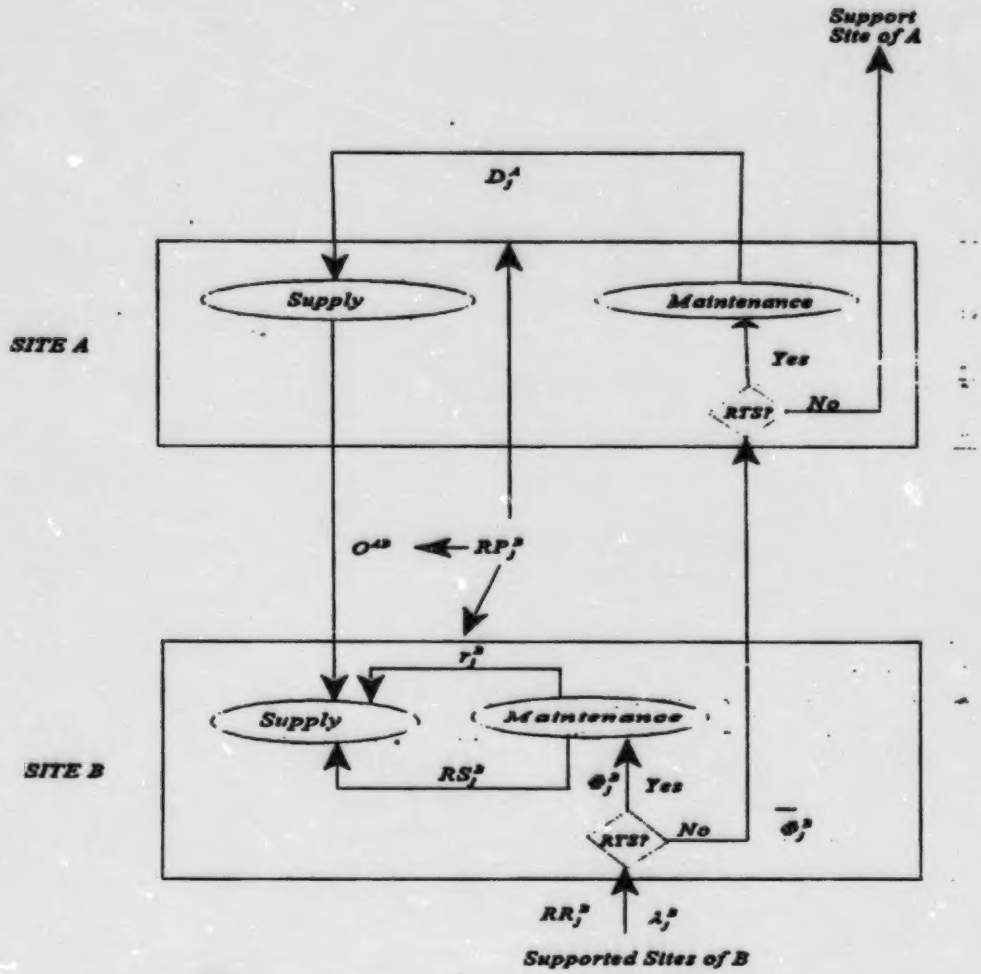

Figure 3: Flow and Time Components to Resupply Part j at Site B

$$D_j^A = \frac{B\left(S_j^A;\ \lambda_j^A\ R_j^A\right)}{\lambda_j^A} \qquad (12)$$

where $B\left(S_j^A;\ \lambda_j^A\ R_j^A\right)$ is the backorder of part j at A which depends not only on the stock level S_j^A but also on the resupply time R_j^A of part j at site A. It can be seen that this delay D_j^A is an amount that varies between zero, when there are no backorders at A, to the resupply time R_j^A at A, when there is no stock there (since $B(0;\lambda_j^A\ R_j^A) = \lambda_j^A\ R_j^A$). Thus it is clear that not only is the resupply time R_j^B a blend of the repair time and the order and ship time from the support site but also of the delay at site A which depends itself on the resupply time R_j^A of part j at A. We see therefore the recursive nature of the resupply times for part j which, in contrast to the demand rates, must begin at the highest echelon level of maintenance.

Expected Resupply Time Caused By The Part Itself

38. This resupply time of item j at site B can best be described by considering three separate cases: site B is below, at, or above the breakdown level of item j. If we are below the breakdown level then the resupply time is simply the order and ship time plus the expected delay , namely, $O_j^{AB} + D_j^A$. If we are at the breakdown level then the resupply time is a weighted average between this last value, if the item is not repaired at B, and the repair time r_j^B if it is. The weighting factor is the fraction ϕ_j^B of failed parts j repaired at site B (and $\bar{\phi}_j^B = 1 - \phi_j^B$ the fraction of those that are not). However we must take into account the fact that at this level we are no longer dealing with the original full assembly but rather with the item without its sub components and we must reduce the failure rate by the application of the type II failure fraction $\bar{f}_j$ for part j (see Equation 10). If we are above the breakdown level the formula would be the same as at the breakdown level except we would not have to modify it with the type II failure fraction as this has already been accounted for. We can summarize these cases in the following table:

Site B is	Resupply time due to part itself
Below breakdown	$O_j^{AB} + D_j^A$
At breakdown	$\bar{f}_j \left(r_j^B \phi_j^B + \left(O_j^{AB} + D_j^A \right) \bar{\phi}_j^B \right)$
Above breakdown	$r_j^B \phi_j^B + \left(O_j^{AB} + D_j^A \right) \bar{\phi}_j^B$

It is easy to see that the following formula summarizes all three cases in one equation:

$$RP_j^B = \Delta_j^B \left(r_j^B \phi_j^B + \left(O^{AB} + D_j^A \right) \bar{\phi}_j^B \right) \tag{13}$$

where Δ_j^B is $\bar{f}_j$ if site B is the breakdown level for item j and is one otherwise, as defined at para 35.

Expected Resupply Time Caused By Subcomponents

39. Here again we can look at what happens below, at and above the breakdown level. Below the breakdown level subcomponents are not yet identified and do not play an explicit role at this point. Above the breakdown level the item has been stripped of its subcomponents and so again they do not play a role. At the breakdown level the resupply time due to a subcomponent k is the average time ρ_k required to replace the failed subcomponent k of item j at site B plus the delay D_k^B in obtaining a working unit at site B. This contribution of each subcomponent must be weighted by its failure fraction f_k. We can sum up the expected resupply time due to subcomponents as:

$$RS_j^B = \delta_j^B \sum_{k=1}^{m} f_k \left(\rho_k + D_k^B \right) \tag{14}$$

where δ_j^B is as defined in para 35.

Average Time to Receive a Broken Item

40. The average time RR_j^B required to receive a broken item j from the supported sites of B is calculated by taking the weighted average of the shipping times from the supported (child) sites. Here we distinguish three cases :

Case 1: This first case identifies the instances when this time element is zero:

a) If site B is a first line site, $RR_j^B = 0$ because there are no supported sites (N=0).

b) If there is no demand for item j at site B, $RR_j^B = 0$ because there is no time being spent waiting for failed items to arrive from supported sites ($\lambda_j^B = 0$).

c) If item j is not broken down at site B and no repairs are made on site then $RR_j^B = 0$ because site B does not have to wait until it receives the item to decide whether it must order a replacement from its support site.

Case 2: If site B is one echelon above the breakdown echelon of item j then the waiting time is the average of the ship time O^{BC} from each of the child sites weighted by the proportion of demand coming in from each child site $\lambda_j^C / \lambda_j^B$. The equation would be

$$RR_j^B = \sum_{C=1}^{N} O^{BC} \frac{\lambda_j^C}{\lambda_j^B} \bar{\phi}_j^C \bar{f}_j \qquad (15a)$$

where the demand for item j at the supported sites must be weighted by the type II repair fraction $\bar{f}_j$ and the percentage of failures shipped to site B for repair, $\bar{\phi}_j^C$.

Case 3: For all remaining cases, we are waiting to receive items that were not repairable at the child site C($\bar{\phi}_j^C$) and are shipped on to site B so that we must add to the ship time O^{BC} the time spent waiting at site C, θ_j^C which is defined recursively below. The equation then becomes:

$$RR_j^B = \sum_{C=1}^{N} \left[\theta_j^C + O^{BC}\right] \frac{\lambda_j^C}{\lambda_j^B} \bar{\phi}_j^C \qquad (15b)$$

We can then summarize these cases in the following equation

$$RR_j^B = \Phi\left(\phi_j^B + \delta_j^B\right) \theta_j^B \qquad (15)$$

where

$$\theta_j^B = \begin{cases} 0 \quad \text{if } \lambda_j^B = 0; \text{ or if } B \text{ is a first line site} \\ \sum_{C=1}^{N} \left[\left(1 - \Phi\left(\phi_j^C + \delta_j^C\right)\right)\theta_j^C + O^{CB}\right] \frac{\lambda_j^C}{\lambda_j^B} \bar{\phi}_j^C \Delta_j^C \end{cases}$$

and where the function

$$\Phi(x) = (1 \text{ if } x > 0;\ 0 \text{ if } x \leq 0) \text{ (Heaviside function)};$$

is used to "switch off" the θ_j^B or θ_j^C according to the above cases. The δ_j^C and Δ_j^C are as defined at para 35.

CALCULATING BACK ORDERS

41. Because the Resupply times are dependent on the back orders at the next higher echelon (equation (13)) and at the next lower indenture level (equation (14)), back orders must be generated hand in hand with the resupply times. In order to simplify the notation we drop the subscript j and superscript B in the following discussions.

42. Assume first that there is only one indenture level, LRUs, so that RS = 0. Also note that RR can be calculated at each site as it is not recursive. RP handles the effects of a multi-echelon case and is a function of the delay of the item at the next higher echelon. To compute equation (13), one must begin with the item at the highest echelon site where, by definition, the delay plus the order and ship time is equal to the procurement lead time of the item. Computing B at this site using equation (8) and applying Little's formula yields the D for this site which can be used to solve for RP at the next lower echelon, and the process is repeated until the resupply times and the back orders of a LRU are calculated at all sites.

43. Assume now that there is only one echelon level, i.e. only one site. RP can be calculated as it depends only on the repair time and procurement lead time. RR again can be calculated without problems. RS handles the effects of a multi-indenture case and is a function of the D of the subcomponent items. To calculate RS for a LRU, one must begin with the lowest indenture item of that LRU where there is no delay for subcomponents (RS=0) and therefore the back orders equation can be solved. Applying Little's formula yields the delay, D, for the item which can then be used to calculate RS for the parent item, and so on, until the backorder for the LRU can be calculated.

44. Therefore to handle the general case, one begins with the lowest indenture items at the highest echelon site where the back orders equation can be computed and then D is determined for all of the lower echelon sites until the breakdown echelon of the parent item has been reached. It is now possible to calculate RS for the parent item but RP may not yet be defined at this echelon. Therefore, one must again begin at the highest echelon site with the next higher idenure items and the process is repeated until the back orders of the LRUs at first line can be calculated.

45. The way a computer program would order the sequencing of the computations would be by categorizing each item by echelon level and indenture level and proceed in steps one through nine as indicated in the following table:

		ECHELON LEVEL					
		6	5	4	3	2	1
INDENT LEVEL	LRU	4	5	6	7	8	9
	SRU	3	4	5	6	7	8
	MOD	2	3	4	5	6	7
	SUBMOD	1	2	3	4	5	6

The order of items within each box and of boxes within each step is unimportant and can be organized for efficient computations. The optimization will focus on the last box in the upper right hand corner which is the last to be computed.

OPTIMIZATION METHOD - THE SINGLE LRU CASE

46. As mentioned earlier, the model considers each LRU separately, therefore, the problem of minimizing the back orders of a single LRU will be explained first before considering the multiple LRU case. In order to measure the decrease in back orders caused by the allocation of a spare, the initial spares distribution is used to establish a baseline. The baseline value is the sum of the back orders of the LRU at 1st line sites which is determined using the method described in the previous section. If there are no initial spares, the baseline value is easily determined from the initial conditions in equation (8) and yields $D = R$, simply the resupply time.

47. Adding an item j at any location B changes the expected number of back orders of that item at that location and ultimately changes the sum of the back orders of the LRU at 1st line sites. This change in total LRU back orders represents the "payoff" of adding item j to the stock at location B and is represented by Δ_j^{B*}, the absolute value of the difference in back orders of first line LRU. This process can be applied to all subcomponents of the LRU at all locations with the payoff results being stored in an item/location matrix. By dividing the payoff by the chosen logistics cost of item j, c_j, the relative back orders per logistics cost can be determined. Logically, the item/location yielding the greatest payoff per logistics cost is chosen to get its stock level increased by one unit and the payoff matrix is updated to take into account the multi-echelon and multi-indenture effects caused by the addition of the spare. Specifically if the item is an LRU at first line then the back orders at that site are recalculated

with a higher stock level. Otherwise the change is calculated by recalculating the resupply times at all sites below B and all indentures above j until the resupply time of the LRU at first line can be computed from which new backorders for the LRU at first line are derived. For instance if a module at third line is identified then the darkened boxes in the following table are recalculated:

ECHELON LEVEL

INDENT LEVEL	6	5	4	3	2	1
LRU	4	5	6	7	8	9
SRU	3	4	5	6	7	8
MOD	2	3	4	5	6	7
SUBMOD	1	2	3	4	5	6

48. The addition of a spare j at location B reduces the effect of a spare j at the support site of B. Therefore, the model next calculates the effect of removing a spare from any location. The increase in total back orders that would result from the removal of a spare is stored in an item/location matrix known as the "penalty" matrix and is represented by Δ_j^{B-}, as the absolute value of the difference in back orders of the LRU at first line. This penalty matrix must also be updated after every spare allocation to take into account the multi-echelon and multi-indenture effects explained above. After the matrices have been updated, a comparison is made between the change in back orders caused by the addition of item j at location B to determine if a removal of an item j at a support site of B should occur. (Sometimes, items already allocated by LOGAN (SPARING) can be removed. This happens when the money recovered in this way yields a greater payoff per dollar if spent for another item/location.) To prevent the model from removing a spare for very little gain, the following formula is used to determine if a removal should occur: if Δ_j^{B+} is the largest value of the payoff matrix and Δ_j^{A-} the largest value of the penalty matrix then

$$\frac{\Delta_j^{A-} - \Delta_j^{B+}}{\Delta_j^{B+}} > 0.1 \qquad (16)$$

A removal actually results in a reallocation and no actual spare is bought in that iteration. As with the addition of a spare, the matrices must be updated if a reallocation occurs.

49. This process of selecting an item/location for stock increase, followed by updating the payoff and penalty matrices, goes on until either the payoff per logistics cost or the total expected back orders for the LRU becomes less than some prescribed value, whichever comes first. These would be set fairly low so as to generate a complete set of trade off values of dollars versus back orders for use in the multiple LRU case.

Increasing the Efficiency of the Algorithm

50. The optimization process begins with the calculation of the change in back orders caused by the addition of a spare j at a specified location B. Since every item of the LRU must be added to every location B, the formulae of equation (8) are applied to efficiently calculate the change in back orders. A change in back orders causes a change in delay time which affects the resupply time and consequently the delay time of any supported sites. This can be a computationally time-consuming task as a change in back orders at a site affects all lower echelon supported sites. Therefore the following approach is used during the optimization process.

51. From our earlier definition of back orders (Eqn 7) and Little's formula.

$$D(S;\lambda R) = \frac{B(S;\lambda R)}{\lambda}$$

$$= \begin{cases} R - \frac{S}{\lambda} - \frac{1}{\lambda}\sum_{k=0}^{S-1} \frac{e^{-\lambda R}(\lambda R)^k}{k!}(k-S), & S \neq 0 \\ \lambda R, & S=0 \end{cases} \qquad (17)$$

Therefore, if S ≠ 0

$$\frac{\partial D}{\partial R} = 1-\frac{1}{\lambda}\left[\sum_{k=0}^{S-1}(-\lambda)\frac{e^{-\lambda R}(\lambda R)^k}{k!}(k-S)+\sum_{k=0}^{S-1}(\lambda k)\frac{e^{-\lambda R}(\lambda R)^{k-1}}{k!}(k-S)\right]$$

$$=1+\sum_{k=0}^{S-1}\frac{e^{-\lambda R}(\lambda R)^k}{k!}(k-S)-\sum_{k=0}^{S-1}\frac{e^{-\lambda R}(\lambda R)^{k-1}}{(k-1)!}((k-1)+1-S)$$

$$=1+\sum_{k=0}^{S-1}\frac{e^{-\lambda R}(\lambda R)^k}{k!}(k-S)-\sum_{k=0}^{S-2}\frac{e^{-\lambda R}(\lambda R)^k}{k!}(k-S)-\sum_{k=0}^{S-2}\frac{e^{-\lambda R}(\lambda R)^k}{k!}$$

$$=1-\frac{e^{-\lambda R}(\lambda R)^{S-1}}{(S-1)!}-\sum_{k=0}^{S-2}\frac{e^{-\lambda R}(\lambda R)^k}{k!}$$

$$=1-\sum_{k=0}^{S-2}\frac{e^{-\lambda R}(\lambda R)^k}{k!}$$

$$=1-A(S-1;\lambda R)$$

so that

$$\frac{\partial D}{\partial R} = \begin{cases} 1-A(S-1;\lambda R) & \text{if } S \neq 0 \\ \lambda & \text{if } S=0 \end{cases} \qquad (18)$$

and if S ≠ 0

$$\frac{\partial^2 D}{\partial^2 R}=-\left[\sum_{k=0}^{S-1}(-\lambda)\frac{e^{-\lambda R}(\lambda R)^k}{k!}+\sum_{k=0}^{S-2}(\lambda k\frac{)^{e^{-\lambda R}(\lambda R)^{k-1}}}{k!}\right]$$

$$=\lambda\left[\sum_{k=0}^{S-1}\frac{e^{-\lambda R}(\lambda R)^k}{k!}-\sum_{k=0}^{S-2}\frac{e^{-\lambda R}(\lambda R)^k}{k!}\right]$$

$$=\lambda\frac{e^{-\lambda R}(\lambda R)^{S-1}}{(S-1)!}$$

$$=\lambda P(S-1;\lambda R)$$

so that

$$\frac{\delta^2 D}{\delta^2 R} = \begin{cases} \lambda\, P(S-1;\lambda R) & \text{if } S \neq 0 \\ 0 & \text{if } S=0 \end{cases} \tag{19}$$

52. A Taylor series expansion of the change in delay time, ΔD, caused by a change in resupply *time*, ΔR, yields

$$\begin{aligned} \Delta D &= D(S,\lambda R) - D(S,\lambda(R-\Delta R)) \\ &= \frac{\partial D}{\partial R}\Delta R - \frac{\partial^2 D}{\partial^2 R}\frac{\Delta R^2}{2!} - \frac{\partial^3 D}{\partial^3 R}\frac{\Delta R^3}{3!} - \ldots \\ &\approx \frac{\partial D}{\partial R}\Delta R - \frac{\partial^2 D}{\partial^2 R}\frac{\Delta R^2}{2!} \end{aligned}$$

Therefore, the following Taylor series approximation is used to estimate the decrease in expected delay, ΔD, when the resupply time decreases by ΔR and S spares are in stock.

$$\Delta D = \begin{cases} (1-A(S-1;\lambda R))\Delta R - \lambda P(S-1;\lambda R)\dfrac{\Delta R^2}{2} & S \neq 0 \\ \lambda \Delta R & S=0 \end{cases} \tag{20}$$

53. Once a spare is chosen, the payoff and penalty matrices are updated using the original formulae rather than the Taylor series approximation to reflect the actual changes caused by the addition of that spare. With the matrices updated, the decision whether to reallocate a spare from a higher echelon is made. If a reallocation occurs, the matrices are once again updated.

HANDLING THE MULTIPLE LRU CASE

54. In practice (see [16]), the total number of back orders for a given type of LRU, when viewed as a function of total cost T, can be approximated by a decreasing exponential function. Hence define

$$f(T) = \begin{cases} \min \sum_B B\left(S_I^B; \lambda_I^B R_I^B\right) \\ \text{subject to } \sum_j \sum_B c_j S_j^B \le T \end{cases}$$

where f is positive decreasing and convex. Now, suppose there are K different types of LRU, then one wants to solve

$$\min\ h(C) = \sum_{j=1}^{K} f_j(C_j) \qquad (20)$$

$$\text{subject to } \sum_{j=1}^{K} C_j \le T, \text{ and } C_j \ge 0$$

where $f_j\ (C_j)$ is an approximation of the back order level caused by the LRU of type j when C_j dollars are optimally spent for it (and its subcomponents) using

$$f_j(C_j) = a_j\, e^{-b_j C_j}$$

The values of a_j and b_j can be estimated by regression analysis on data points obtained through the single LRU case above evaluated at several appropriate values of C_j.

55. One wants to determine how to 'partition' the global budget T between the K types of LRUs. That is, to determine the C_j's. In order to find necessary and sufficient conditions for an optimal partition of T, the Kukn-Tucker conditions are applied to (20) as demonstrated in Ref [9].

$$\text{Let } \beta(C,\lambda) = h(C) - \lambda_1(T - \Sigma C_j) - \lambda_2 C_1 - \ldots - \lambda_{k+1} C_K$$

then the Kuhn-Tucker conditions are

$$\nabla_c \beta(C,\lambda) = 0$$

$$T - \sum C_j \geq 0$$

$$C_j \lambda_i \geq 0$$

$$\lambda_1 \left(T - \sum C_j\right) = 0$$

$$\lambda_{j+1} C_j = 0.$$

with

$$\nabla_c \beta(C,\lambda) = \begin{pmatrix} \frac{df_1}{dC_1} + \lambda_1 - \lambda_2 \\ \frac{df_2}{dC_2} + \lambda_1 - \lambda_3 \\ \vdots \\ \frac{df_K}{dC_K} + \lambda_1 - \lambda_{K+1} \end{pmatrix}$$

Let C^* and λ^* be the solution to the Kuhn-Tucker conditions (the solution is unique since h(C) in (20) is a strictly convex function being the sum of strictly convex functions); then for $C_j^* > 0$, one has $\lambda_{j+1}^* = 0$, j=1, ..., k.

Hence

$$\frac{df_m(C_m^*)}{dC_m} = -\lambda_1^* = \frac{df_n(C_n^*)}{dC_n} \quad \text{if } C_m^*, C_n^* > 0$$

Let I be the set of indices for which $C_i^* > 0$, and E is the set of indices for which $C_i^* = 0$. Then a partition $C_1^*, C_2^*, \ldots, C_K^*$ of the budget T is optimal if and only if $\sum C_j^* = T$, $\exists \lambda_1 > 0$ such that

$$\frac{df_m(C_m^*)}{dC_m} = \frac{df_n(C_n^*)}{dC_n} = -\lambda_1 \text{ for } \forall m,n \in I,$$

and $-\frac{df_p(0)}{dC_p} \le \lambda_1$ for $p \in E$. In our case, the derivatives df_j/dC_j are negative increasing functions of C_j. Specifically

$$\frac{-df_j(C_j^*)}{dC_j} = \begin{cases} a_j b_j e^{-bf_j^*} = \lambda_1^* & j \in I \\ a_j b_j < \lambda_1^* & j \in E \end{cases}$$

From

$$T = \sum_I C_j^* = \sum_J -\frac{1}{b_j} \ln\left(\frac{\lambda_1^*}{a_j b_j}\right) = \sum_I \frac{\ln(a_j b_j)}{b_j} - \left(\sum_I \frac{1}{b_j}\right) \ln\lambda_1^*$$

we obtain

$$\lambda_1^* = \exp\left(\left(\sum_I \frac{\ln(a_j b_j)}{b_j} - T\right) \Big/ \sum_I \frac{1}{b_j}\right)$$

$$C_j^* = \begin{cases} \frac{1}{b_j} \ln\left(\frac{a_j b_j}{\lambda_1^*}\right) & j \in I \\ 0 & j \in E \end{cases} \qquad (21)$$

56. If we wished to specify a backorder target B instead and find the minimum cost to attain the backorder target, then we would solve

$$\min \sum_{j=1}^{K} C_j \qquad (22)$$

$$\textit{subject to } h(C) \leq B$$

Again applying the Kuhn-Tucker conditions to the Lagrangian of (21)

$$L(C,\lambda) = \sum_{j=1}^{K} C_j - \lambda_1 (B - h(C)) - \lambda_2 C_1 - \cdots - \lambda_{K+1} C_K$$

we have

$$\nabla_C L = \begin{pmatrix} 1+\lambda_1 \dfrac{df_1}{dC_1} - \lambda_2 \\ \vdots \\ 1+\lambda_1 \dfrac{df_K}{dC_K} - \lambda_{K+1} \end{pmatrix} = 0$$

$$B - \sum_{j=1}^{K} f_j(C_j) \geq 0$$

$$C_j \, , \, \lambda_j \geq 0$$

$$\lambda_1 (B - \sum_{j=1}^{K} f_j(C_j)) = 0$$

$$\lambda_{j+1} C_j = 0$$

With I and E defined as above and optimal solution C^*, λ_j^* we have (note that as above a unique solution exists and if $\lambda_1^* = 0$ then $I = \phi$ and only the trivial solution exists)

$$-\frac{df_j(C_j^*)}{dC_j} = \begin{cases} a_j b_j e^{-b_j C_j^*} = \frac{1}{\lambda_1^*} & \forall\, j \in I \\ a_j b_j < \frac{1}{\lambda_1^*} & \forall\, j \in E \end{cases}$$

$$B = \sum_I f_j(C_j^*) = \sum_I a_j e^{-b_j C_j^*} = \frac{1}{\lambda_1^*} \sum_I \frac{1}{b_j}$$

so that

$$\lambda_1^* = \frac{1}{B} \sum_I \frac{1}{b_j}$$

$$C_j^* = \begin{cases} \frac{1}{b_j} \ln a_j b_j \lambda_1^* & j \in I \\ 0 & j \in E \end{cases} \qquad (23)$$

57. Thus in either problem (20) or (22) we first compute the λ_1^* as in (21) or (23) then determine E as those j for which $-\frac{df_j(0)}{dC_j} \leq \lambda_1^{*(n)}$, setting those $C_j^* = 0$ and computing the others as in (21) or (23) respectively. Thus for each LRU j we have a target C_j^* up to which we can allocate as in the previous section.

TAKING EXISTING STOCK INTO ACCOUNT

Previously Distributed Stock

58. The model uses previously distributed stock to set the baseline value to be used in the allocation process and will not permit removal of these stock from their assigned location during the allocation of any further stock acquisition. This can be achieved by setting Δ_j^{B-} to infinity (i.e. a sufficiently large number) whenever j is at the "previously distributed level" at site B. In essence this sets minimal stock levels of each item at each location. Though the current LOGAN (Sparing) does not explicitly have a feature to set minimum levels, it could be introduced in the same fashion. A feature to validate minimum levels as feasible against the specified target (budget/backorder) would be required.

Existing Undistributed Stock .

59. One of the advantages of this allocation method is that it can optimally distribute existing stocks before adding any new items. This is a very useful feature when spare provisioning is phased or reassessing existing allocations after parameter changes during the in-service period. This subsection describes how this distribution is achieved.

60. Let qty_j be the quantity of existing undistributed stock for item j, and $dqty_j$ be the quantity of item j distributed so far by the allocation process. Recall that c_j represents the cost of item j, and let $\tilde{c}_j$ represent the temporary cost of item j being considered during the allocation process. Before starting the allocation process, $dqty_j$ is set to 0 for all j, and the temporary cost, $\tilde{c}_j$, is set to c_j for all j. As the allocation of spares proceeds, the values of $dqty_j$'s increase. Eventually, the existing stock for item j will be distributed and $dqty_j = qty_j$. At this point, for all B, Δ_j^{B-} is set to infinity, and the temporary cost, $\tilde{c}_j$, is also set to infinity. This has the effect of stopping any further additions of item j since the payoff $\Delta_j^{B+} / \tilde{c}_j$ is zero for all B. It also has the effect of preventing any removal of item j since Δ_j^{B-} / c_j is too large. Therefore, only the stock levels for the other items are allowed to change. As under previously distributed stock this methodology could be implemented for setting maximum levels of stock, a feature not explicitly implemented in LOGAN (Sparing).

61. At some point, one obtains $dqty_j = qty_j$ for all j which implies that the existing stocks have been distributed. At this point, new items can be added by resetting $\tilde{c}_j$ to c_j for all j. Note that in order to preserve the relationship $dqty_j \geq qty_j$ for all j thereafter, Δ_j^{B-} should be set to infinity whenever $dqty_j$ becomes equal to qty_j as was the case in the previous section.

MEASURES OF EFFECTIVENESS

Back orders

62. As defined earlier, a back order occurs whenever a maintenance location is unable to satisfy a need for a spare, and it was shown (equation (7)) that:

$$B(S;\lambda R) = \begin{cases} \lambda R - S - \sum_{k=0}^{S-1} \frac{e^{-\lambda R}(\lambda R)^k}{k!}(k-S), & S \neq 0 \\ \lambda R, & S=0 \end{cases}$$

This formula is used to determine the expected number of back orders for each LRU at a 1st line site. The LRU back orders are summed over the M LRUs to arrive at the expected LRU back orders for the PE at each 1st line site B, and then the expected back orders for all the PE is calculated by taking the average of the LRU back orders over all 1st line sites, namely

$$B = \sum_{B=1}^{N} \frac{1}{N} \sum_{j=1}^{M} B\left(S_j^B; \lambda_j^B R_j^B\right)$$

Delay Time

63. Delay time is derived from Little's formula and can be defined as the expected time required to obtain a working item at a particular location when a demand occurs. The formula (equation (12)) used in the model is:

$$D_j^B = \frac{B\left(s_j^B;\ \lambda_j^B R_j^B\right)}{\lambda_j^B} \quad \ldots \quad \ldots \quad \ldots$$

This formula is used to determine the expected supply delay for each LRU at a 1st line site. The expected supply delay for the PE at a 1st line site is calculated by taking the weighted average of the supply delays for each LRU at that site. Then the expected supply delay for all the PE is calculated by taking the average of the expected supply delays for the PE over all 1st line sites. If the failure rates at each first line site are all the same or very close to each other this measure is essentially the same as the backorder measure rescaled by the common failure rate.

Continuous Availability

64. LOGAN (Sparing), being interested in MOE's only insofar as they impact on sparing, does not consider within its availability measure issues of hardware reliability and maintainability. In particular it excludes preventive maintenance delays. Under such considerations for continuous availability the system may be in one of two states operating (up) and not ready to operate (down). Before proceeding further the following definitions are standard:

$MTBM$ = *mean time between maintenance*
MDT = *mean down time due to spares and corrective actions*
$MTBF$ = *mean time between failures*

In this scenario a failure leads to down time followed by up time followed by failure. The MTBM represents then the up time (see Figure 4) and we have in this case (see [17] page 46)

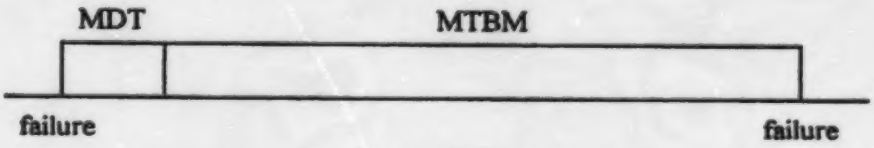

Figure 4: Failure Cycle

the following approximation:

$$MTBM \approx MTBF \tag{24}$$

The standard definition of operation availability (see [5], [17], [10]) is:

$$A_0 = \frac{MTBM}{MTBM + MDT} \tag{25}$$

which can be rewritten, using (24), as

$$A_0 = \frac{1}{\left(1 + \frac{MDT}{MTBF}\right)} \tag{26}$$

65. Consider base B at which operate E^B equipments each composed of M LRU's with exponential failure rates so that the overall process is exponential with the failure rate the sum of the failure rates of the LRUs. For any individual equipment

$$\lambda^B = \sum_{j=1}^{M} \frac{\lambda_j^B}{E^B}$$

and

$$MDT = \sum_{j=1}^{M} \left(D_j^B + \rho_j\right) \frac{\lambda_j^B}{\lambda^B E^B}$$

$$MTBF = \frac{1}{\lambda^B}$$

that is MDT is the weighted average of the delay (supply delay and replacement) times for each LRU, weighted by the failure rates. Thus the operational availability of any individual equipment at base B is

$$A_0^B = \frac{1}{1 + \sum_{j=1}^{M} \left(D_j^B + \rho_j\right) \frac{\lambda_j^B}{E^B}} \tag{27}$$

66. For the system we average over the N first line sites to obtain

$$A_0 = \frac{1}{N} \sum_{B=1}^{N} \left(\frac{1}{1 + \sum_{j=1}^{M} \left(D_j^B + \rho_j\right) \frac{\lambda_j^B}{E^B}} \right) \tag{28}$$

As an objective function which is not separable in terms of the LRUs, it cannot be used directly in the marginal analysis process to maximize availability. In order to circumvent this difficulty the following approximation is used:

$$A_R = \frac{1}{1+\frac{1}{N}\sum_{j=1}^{M}\sum_{B=1}^{N}\left(D_j^B + \rho_j\right)\frac{\lambda_j^B}{E^B}} \tag{29}$$

Thus we can minimize $\frac{1}{A_R}$ which is separable.

67. The only notes documenting this approximate measure are reproduced in Annex A. Unfortunately the justification is very weak. The expansions into geometric series $\frac{1}{1+a} = 1 - a + a^2 - a^3 + \cdots$ is valid if and only if $|a| < 1$, namely $\frac{MDT}{MTBF}$ or the sum of these at all first line sites is less than one. There is no guarantee of this (see example in [10]) and, even if it were true, values closer to 1 would cause slow convergence and not guarantee similar answers on second order terms. In order to properly justify this approximation an interval [0,L] would have to be established such that

$$\frac{|A_0 - A_R|}{A_R} < .05$$

Other authors (for instance [5]) use different approaches. A thorough review of the literature and further work are needed to identify and assess different options that could be usable within the context of LOGAN. Since some form of approximation is required any alternative would have to be validated for error against (28) as stated above.

Intermittent Availability

68. An article by Seger [10] noted that the formula used above for continuous availability yields incorrect results for systems employed intermittently with a fixed operating requirement because it does not account for the time when the PE is idle but ready to operate. Thus if TH is the total hours in a period, OH the operating hours in that period and DH the down

hours in that period, the ready hours RH would be

$$RH = TH - OH - DH$$

The continuous case is actually included here when $RH = 0$. In [10] it is shown that

$$A_I = 1 - \frac{MDT}{(K \times MTBM)}$$

where $K = \frac{TH}{OH}$. Caution is required with this formula however.

Since $DH = MDT \times \frac{OH}{MTBM}$ we have

$$1 + \frac{MDT}{MTBM} = \frac{1}{OH}(OH + DH) \leq \frac{TH}{OH} = K. \qquad (28)$$

This condition must be validated as MDT is a consequence of sparing and OH is set independently, that is, if OH is set too high for sparing delays and the required maintenance actions to cope with, then availability will become negative indicating an infeasible situation. Using equation (24) we can write

$$A_0 = 1 - \frac{1}{K}\frac{MDT}{MTBF}$$

69. Consider base B at which operate E^B equipments each composed of M LRU's, as in para 65. The intermittent availability of any individual equipment at base B is

$$A_I^B = 1 - \frac{1}{K^B}\sum_{j=1}^{M}\left(D_j^B + \rho_j\right)\frac{\lambda_j^B}{E^B}$$

where $K^B = \frac{TH}{OH^B}$ and OH^B are the operating hours at base B.

70. For the system we average over the N first line sites to obtain

$$A_I = \frac{1}{N}\sum_{B=1}^{N} A_0^B = 1 - \frac{1}{N}\sum_{B=1}^{N}\frac{1}{K^B}\sum_{j=1}^{M}\left(D_j^B + \rho_j\right)\frac{\lambda_j^B}{E^B} \qquad (29)$$

As an objective function, it remains separable in terms of the LRU's and could be used directly in the marginal analysis process to maximize availability. The relation to the back orders can be seen by replacing the supply delay time by the back order quantity using (12), to yield

$$A_I = 1 - \frac{1}{N}\sum_{B=1}^{N}\sum_{j=1}^{M}\frac{B\left(S_j^B;\lambda_j^B R_j^B\right)}{K^B E^B} - \frac{1}{N}\sum_{B=1}^{N}\sum_{j=1}^{M}\frac{\rho_j\lambda_j^B}{K^B E^B}$$

In particular if the same number of equipments operate the same number of operating hours per period at each first line site then

$$A_I = 1 - \frac{1}{KE}\left(B + \sum_{j=1}^{M}\rho_j\,\lambda_j\right)$$

where KE represents the common factor $K^B E^B$ and λ_j the common base failure rate of item j. In general this A_0 weights back orders according to the level of activity at the first line sites.

Probability Of Mission Accomplishment

71. In many problems that arise in practice it is clear that item failures will have variable impact upon the capability of a piece of equipment to perform its mission. Factors such as sub-system redundancy, criticality of item failure and the mission profile all play a role in determining the impact of an item failure, and the response of the supply and maintenance organizations to it. Such issues are particularly important for isolated spares inventories with limited storage capabilities. For example, such situations occur when ships are at sea or for isolated sites with restricted support capabilities.

72. In considering such issues the original METRIC models utilize essentialities, e_j^B, or relative back order costs of item j at site B. These are interpreted as weighting functions employed during the marginal allocation:

$$\min \sum_{j=1}^{M} \sum_{B=1}^{N} e_j^B B\left(S_j^B; \lambda_j^B R_j^B\right) \quad \ni \quad \sum_{j=1}^{m} \sum_{B=1}^{n} S_j^B C_j \leq C$$

Essentialities represent the relative cost of a back order compared to a back order of some standard item. Mod-METRIC like models treat essentiality as equal across all items and all locations. By definition essentialities may only be clearly defined for first indentured items.

73. Under both models repair of a failed item is initiated immediately upon failure and the item is replaced, if it is available at local supply. In some environments such a failure response does not occur. Replacement of items deemed non-critical may be ignored until a more convenient opportunity presents itself, or the item may be removed and the system operated without the failed unit. Further, systems with built in redundancy may be left until the equipment fails completely, with a repair action taken on all the failed items in the system. Such factors determine the response given to a failure. However, essentialities do not reflect changes in the repair time that result from the failure of mission critical items.

74. Most standard models ignore the impact of such "atypical" failure responses. The inherent assumption is that the weapon system is described as a series reliability system. If any item fails the equipment itself has failed. For many situations this is a normal consequence of peace-time operating procedures.

75. The interpretation of item essentialities as weighting functions within the optimization methodology presents a number of drawbacks. First, the true measure of the objective function is distorted in the process. Second, little justification seems to be provided for deciding on the values of these relative weights. Third, such weightings must be modified for each choice of objective function. Fourth, the impact of sub-items failure on the weapon system is considered within such a framework, only in so far as they affect indirectly the first indentured items. Finally, no clear framework is in place to consider sub-system redundancy or properly reflect mission requirements.

76. Given these problems it was apparent that easy interpretation of item essentialities and the resulting modified objective functions would be difficult for most end-users to interpret. Therefore an alternate methodology is utilized within LOGAN (Sparing).

77. The central question in deciding relative impact of failures is "Given that a failure has occurred, what is the impact on the prime equipment?". Since, all failures are traceable to type II failures we need only consider the impact of type II failures in the system. The terms critical or essential failures will be used to signify that the prime equipment is unable to complete its mission given a failure in the item.

78. Given a type II failure, let β be the conditional probability that the failure is considered critical. We shall refer to this as the Probability of Mission Impact (PMI) and knowledge of β is assumed to be exact. Given j such failures in the system the probability that i are critical is binomial

$$b(j;i,\beta) = \binom{j}{i} \beta^i (1-\beta)^{j-i}$$

Given an arbitrary demand distribution P_k for k units over a certain period of time, with mean λ, the probability that we experience i critical failures is P_i', determined by the convolution of the two distributions

$$P_i' = \sum_{k=i}^{\infty} P_k b(k;i,\beta)$$

By comparing the associated characteristic functions it may be shown that P_i' is the same distribution as P_k with the mean and variance-to-mean ratios given by

$$\lambda' = \beta\lambda: \quad \textit{expected demand}$$
$$V' = 1+\beta(V-1): \quad \textit{mean-to-variance ratio.}$$

This last result is important as it implies that modifications to existing sparing algorithms do not require adjustment to the underlying demand distribution proper. In order to consider critical demand only the mean and the variance associated with an item need be modified. In particular since we assume Poisson demand the critical demands are Poisson with a lower demand rate $\beta\lambda$.

79. From this expression and equation (12) it is clear that if the over-all supply delay is insensitive to changes to the demand rate then backorders are reduced by the same factor β. This provides some justification for the traditional utilization of essentialities. However, it is clear that using essentialities as weights will not properly consider the modified variance for non-Poisson demand.

80. Information on the PMI is to be found as part of the FMECA (Failure Mode, Effects and Criticality Assessment), along with multipliers to the demand rate for mission type, environmental factors, duty cycles, etc. Hence, the necessary information is available on a per failure mode basis. Determination of the PMI may then be made by averaging over all projected missions, environments, etc. Mission specific parameters enter strictly as multipliers on the demand rate.

81. Failure data is typically provided on the basis of the aggregate failure modes of the item i.e. its type I failure modes. In order to consider this properly the aggregate conditional probability of mission impact must be determined for each part j. This may be done by the following simple recursive relationship:

$$\beta_j = \sum_{k=1}^{m} f_k \beta_{jk} + B_{j0}\left(1 - \sum_{k=1}^{m} f_k\right) \tag{30}$$

where

m = *number of subcomponents of part j*;

f_k = *type I failure fraction of subcomponent k of part j* (*see Equation* (9));

β_{jk} = *aggregate PMI for subcomponent k of part j*

β_{j0} = *PMI for a Type II failure of part j.*

The recursion can start at the lowest indenture level where only Type II failures occur (i.e. $f_k = 0 \quad \forall k$). The result is to roll-up the influence each sub-component has on it's associated higher assemblies. Therefore the impact of the essentiality measure β_j is well defined for all sub-items.

82. The impact of critical failures is felt not only on the demand rate but also on the response of the support system to non-critical demands. In order to consider this issue we make the following simplifying assumption: All non-critically failed items are repaired immediately in a fashion consistent with the uninterrupted operation of the prime equipment and do not influence repair times for critical failures. This assumption holds only for first line locations. Thus, for purposes of modelling we separate critical and non-critical response at first-line. Second and higher echelon lines of repair and support see the true demand for spares.

V. CONCLUSION

83. This review of the mathematics underlying LOGAN (Sparing) was based on a synthesis of various documents produced over the several years of the model's evolution. The lack of a formal configuration management approach in this development has left a poor collection of documentation. These documents, some of which were formal reports, others simply notes that were found, are in general neither complementary nor complete. Where gaps in explanation were found to exist, these were infilled with the development of the necessary theory and equations. Explanations that are in conflict or are insufficiently substantiated were highlighted for further work. This lack of meticulous version management and documentation over the years raises a number of reliability issues which must be addressed. Among remedial actions the following must be investigated immediately.

a. A complete audit and rationalization of the code must be performed against the theory identified in this report.

b. Basic assumptions underlying the multiple LRU case were extracted from [16], a study conducted for the METRIC case (i.e. 2-echelon, 1 indenture problem). No explanation of the validity that the assumptions in [16] are transferable to the multi-indenture multi echelon case was given. Though it appears to be valid, this must be verified.

c. The availability formulae need reviewing. The continuous case uses an approximation whose accuracy is not established and differs significantly from those of other authors. The condition of (28) needs to be checked and the code should be verified for this check, as well as for the proper use of the availability measure (if (28) is a strict inequality then A_i must be used). None of the formulae account for preventive maintenance actions or whether demand is measured in calendar time or operating time or can account for usage in other forms, such as kilometers. The availability formulae need to be rationalized.

d. The only documentation for the Probability of Mission accomplishment section were notes left by a Defence Scientist which could not be completely explained. In particular equation (30) as rationalized here is different from the notes found and should be checked with the code. This capability should also be reviewed for pertinence to current users.

84. Other areas that need addressing are the demand modelling with the Poisson and the ability to deal with common items. The author of [5] discusses methodologies to improve demand prediction and to account for common items. These should be investigated for viability in LOGAN (Sparing).

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ANNEX A
PROJECT REPORT PR-9707
JUNE 1997

APPROXIMATING CONTINUOUS AVAILABILITY

1. As we have noted LOGAN (Sparing) utilizes the separable nature of the optimization problem to consider each LRU independently, rolling up the associated distributions at the end. However, difficulties are encountered with this approach when considering operational availability.

2. Consider a series reliability system composed of M independent exponentially distributed components (LRUs). For such a system the demand distribution is Poisson. As such, the steady state operational availability (equation 9) averaged over all locations is given by

$$A_0 = \frac{1}{N}\sum_{B=1}^{N}\left(\frac{1}{1+\sum_{j=1}^{M}\left(D_j^B+\rho_j\right)\frac{\lambda_j^B}{N^B}}\right) \qquad \text{(A1)}$$

It should be noted that equation (A1) is valid for a first-line Poisson demand process under the assumption that simultaneous failures may not occur (i.e. no demands are generated by the remaining items during the down time).

3. Difficulties arise within LOGAN (Sparing) in using equation (A1) as LOGAN (Sparing) first calculates the optimal solution for each LRU. Since equation (A1) requires knowledge of all LRUs at all locations in order to be evaluated it is difficult to evaluate the change in availability given a change in the distribution of any particular LRU.

4. In order to circumvent this difficulty LOGAN (Sparing) utilizes the following approximation to operational availability during the optimization process:

$$A_0 \approx A_R = \frac{1}{1+\frac{1}{N}\sum_{j=1}^{M}\sum_{B=1}^{N}\left(D_j^B + \rho_j\right)\frac{\lambda_j^B}{E^B}} \qquad \text{(A2)}$$

Equations (A1) and (A2) are equal given that the availabilities are equal across bases.

5. In order to demonstrate the equivalence it is instructive to expand equations (A1) and (A2) in a binomial expansion. In order to simplify the resulting expressions define

$$w_j^B = \left(D_j^B + \rho_j\right)\frac{\lambda_j^B}{E^B}$$

Expanding equation (A1) yields

$$A_0 = \frac{1}{N}\sum_{B=1}^{N}\left[1-\left(\sum_{j=1}^{M}\omega_j^B\right)+\left(\sum_{j=1}^{M}\omega_j^B\right)^2-\ldots\right]$$

and equation (A2)

$$A_R = 1-\frac{1}{N}\sum_{B=1}^{N}\sum_{j=1}^{M}\omega_j^B+\left(\frac{1}{N}\sum_{B=1}^{N}\sum_{j=1}^{M}\omega_j^B\right)^2-\ldots \qquad \text{(A3)}$$

It is obvious from the form of equation (A3) that the first two elements may be determined by summing over the locations first, and the LRU's second. This is consistent with the optimization methodology utilized within LOGAN(Sparing). Hence the first two terms are identical in the expansion. However, this is definitely not the case with second and higher order terms. The difference between the two expressions may be presented as

$$A_0 - A_R = \left[\frac{1}{N} \sum_{B=1}^{N} \left(\sum_{j=1}^{M} \omega_j^B \right)^2 - \frac{1}{N^2} \left(\sum_{B=1}^{N} \sum_{j=1}^{M} \omega_j^B \right)^2 + O\left(\left[\omega_j^B \right]^3 \right) \right]$$

$$= \frac{1}{N} \sum_{B=1}^{N} \sum_{j=1}^{M} \sum_{k=1}^{M} \left(\omega_j^B \omega_k^B - \frac{1}{N} \sum_{C=1}^{N} \omega_j^B \omega_k^C \right) + O\left(\left[\omega_j^B \right]^3 \right)$$

given that the two definitions of availability are equal to first order. For higher orders, it is seen that the two definitions are equal if, at the very least the operational availability of the prime equipment is nearly constant across locations. More precisely if the delay times and arrival rates per LRU are constant across locations then (A1) and (A2) are equal.

6. While generating the final output reports, Equation (A2) is used to define the availability of an LRU at a 1st line site. The continuous availability for the PE at a 1st line site is calculated with the same formula but instead of using the MTBF and supply delay of the item, an MTBF for the PE and an expected supply delay for the site are calculated by taking the weighted average of all the LRU MTBFs within the PE and the supply delays for each LRU at that site. The continuous availability of all the PE is calculated by taking the average of the availability determined for each 1st line site.

DOCUMENT CONTROL DATA
(Security classification of title, body of abstract and indexing annotation must be entered when the overall document is classified)

1. ORIGINATOR (the name and address of the organization preparing the document. Organizations for whom the document was prepared e.g. Establishment Sponsoring a contractor's report, or tasking agency, are entered in Section 8).

Operational Research Division
Department of National Defence
Ottawa, Ontario K1A 0K2

2. SECURITY CLASSIFICATION (overall security classification of the document, including special warning terms if applicable)

UNCLASSIFIED

3. TITLE (the complete document title as indicated on the title page. Its classification should be indicated by the appropriate abbreviation (S, C or U) in parentheses after the title)

LOGAN (SPARING) V1.0: MATHEMATICAL THEORY (Release 1.0)

4. AUTHORS (last name, first name, middle initial)

Vincent, Dr. P.A.; Ormrod, M.K.; and Willner, L.

5. DATE OF PUBLICATION (month Year of Publication of document)	6a. NO OF PAGES (total containing information. Include Annexes, Appendices, etc.)	6b. NO OF REFS (total cited in document)
June 1997	64	17

7. DESCRIPTIVE NOTES (the category of document, e.g. technical report, technical note or memorandum. If appropriate, enter the type of report e.g. interim, progress, summary, annual or final. Give the inclusive dates when a specific reporting period is covered.)

Project Report

8. SPONSORING ACTIVITY (the name of the department project office or laboratory sponsoring the research and development. Include the address).

D Log A

9a. PROJECT OR GRANT NO. (if appropriate, the applicable research and development project or grant number under which the document was written. Please specify whether project or grant.)	9b. CONTRACT NO. (if appropriate, the applicable number under which the document was written.)
3561-MOD-SAM	---

10a. ORIGINATOR's document number (the official document number by which the document is identified by the originating activity. This number must be unique to this document.)	10b. OTHER DOCUMENT NOS. (Any other numbers which may be assigned this document either by the originator or by the sponsor.)
ORD Project Report PR-9707	---

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The Directorate of Logistics Analysis (D Log A) has developed a new computer model known as the LOGistics ANalyzer (LOGAN) that integrates three Logistic Support Analysis (LSA) functions; sparing, level of repair analysis, and life cycle costing under a single shell. The SPARING Module is primarily designed to determine the optimal allocation and quantity of repairable spares within a Maintenance Organization to achieve a desired system goal at minimum cost. The Module can also be used to allocate or reallocate existing stock or to determine what additional spares are required in a reprovisioning exercise. The primary objective of this document is to provide a mathematical description of the SPARING Module of the LOGAN program for those who require an in-depth knowledge of the model. The methodology, assumptions, and limitations of the model are described.

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